e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 2

Subject:Mathematics

Title/Heading of topic: Leibnitz's theorem

By Dr. Hari kant singh

Associate professor in mathematics

Rrs college mokama patna

## 1.2 LEIBNITZ'S THEOREM

If $u$ and $v$ are functions of $x$ such that their $n^{th}$ derivatives exist, then the $n^{th}$ derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where $u_r$ and $v_r$ represent $r^{th}$ derivatives of $u$ and $v$ respectively.

**Example11** Find the $n^{th}$ derivative of $x \log x$

**Solution:** Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n} x + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}[-(n-1)+n]$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

**Example 12** Find the $n^{th}$ derivative of $x^2 e^{3x}\sin 4x$

**Solution:** Let $u = e^{3x}\sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\,25^{\frac{n}{2}}\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$

$$= e^{3x}\,5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x}\sin 4x\right)_n = x^2 e^{3x}\,5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) +$$

$$2nxe^{3x}\,5^{n-1}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) +$$

$$n(n-1)e^{3x}\,5^{n-2}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right) + 0$$

$$= e^{3x} 5^n \left[x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

**Example 13** If $y = a\cos(\log x) + b\, sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

**Solution:** Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\,cos(logx)$$

Differentiating both sides w.r.t. $x$ , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\, sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{a\cos(logx) + b\, sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + (y_{n+1}x + n_{C_1}y_n.1) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

**Example 14** If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

**Solution:** $y = \log(x + \sqrt{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. $x$, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$[y_{n+2}(1+x^2)+n_{C_1}y_{n+1}2x+n_{C_2}y_n.2]+\left(y_{n+1}x+n_{C_1}y_n.1\right)=0$$

$$\Rightarrow y_{n+2}(1+x^2)+y_{n+1}2nx+n(n-1)y_n+y_{n+1}x+ny_n=0$$

$$\Rightarrow (1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$$

**Example 15** If $y = \sin\,(m\,sin^{-1}x)$, show that

$(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n$. Also find $y_n(0)$

**Solution:** Here $y=\sin(m\,sin^{-1}x)$ ......①

$$\Rightarrow y_1=\frac{m}{\sqrt{1-x^2}}\cos(m\,sin^{-1}x) \quad ......②$$

$$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2\cos^2(m\,sin^{-1}x)$$

$$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2[1-\sin^2(m\,sin^{-1}x)]$$

$$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)......③$$

$$\Rightarrow (1\text{-}x^2){y_1}^2+m^2y^2=m^2$$

Differentiating w.r.t. $x$, we get

$$(1-x^2)2y_1y_2+{y_1}^2(-2x)+m^2 2yy_1=0$$

$$\Rightarrow (1-x^2)y_2-xy_1+m^2y=0$$

Using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2)+n_{C_1}y_{n+1}(-2x)+n_{C_2}y_n(-2)]-\left(y_{n+1}x+n_{C_1}y_n1\right)+m^2y_n=0$$

$$\Rightarrow y_{n+2}(1-x^2)-y_{n+1}2nx-n(n-1)y_n-(y_{n+1}x+ny_n)+m^2y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n......④$$

Putting $x=0$ in ①, ②and ③

$y(0)= 0, y_1(0)=m$ and $y_2(0)=0$

Putting $x=0$ in ④

$y_{n+2}(0)=(n^2-m^2)y_n(0)$

Putting $n=1,2,3$ ... ... in the above equation, we get

$y_3(0)=(1^2-m^2)y_1(0)$

$=(1^2-m^2)m \quad \because y_1(0)=m$

$$y_4(0) = (2^2 - m^2)y_2(0)$$

$$= 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = \left(3^2 - m^2\right) y_3(0)$$

$$= m(1^2 - m^2)(3^2 - m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0\,, & if\ n\ is\ even \\ m(1^2 - m^2)(3^2 - m^2) \ldots [(n-2)^2) - m^2]\,, & if\ n\ is\ odd \end{cases}$$

**Example 16** If $y = e^{msin^{-1}x}$, show that
$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also find $y_n(0)$.

**Solution:** Here $y = e^{msin^{-1}x}$ ...①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}} \mathrm{e}^{m\, sin^{-1}x}$$

$$= \frac{my}{\sqrt{1-x^2}} \quad \ldots\ldots ②$$

$$\Rightarrow (1 - x^2){y_1}^2 = m^2 y^2$$

Differentiating above equation w.r.t. $x$, we get

$$(1 - x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$$

$$\Rightarrow (1 - x^2)y_2 - xy_1 - m^2 y = 0 \quad \ldots\ldots ③$$

Differentiating above equation $n$ times w.r.t. $x$ using Leibnitz's theorem, we get

$$[y_{n+2}(1 - x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n1\right) - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1 - x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2y_n = 0$$

$$\Rightarrow (1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③

$y(0) = 1,\ y_1(0) = m$ and $y_2(0) = m^2$

Also putting $x = 0$ in , we get

$$y_{n+2}(0) = (n^2 + m^2)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$y_3(0) = (1^2 + m^2)y_1(0)$

$=(1^2 + m^2)m \qquad \because y_1(0) = m$

$y_4(0) = (2^2 + m^2)y_2(0)$

$= m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$

$y_5(0) = \left(3^2 + m^2\right)y_3(0)$

$= m(1^2 + m^2)(3^2 + m^2)$

$\vdots$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2 + m^2) \ldots [(n-2)^2 + m^2], & if\ n\ is\ even \\ m(1^2 + m^2)(3^2 + m^2) \ldots [(n-2)^2 + m^2], & if\ n\ is\ odd \end{cases}$$

**Example 17** If $y = tan^{-1}x$ , show that

$(1 - x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ . Also find $y_n(0)$

**Solution:** Here $y = tan^{-1}x$.....①

$\Rightarrow y_1 = \frac{1}{1+x^2}$ ......②

$y_2 = \frac{-2x}{1+x^2}$

$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 0$ ......③

Differentiating equation ③ $n$ times w.r.t. $x$ using Leibnitz's theorem

$[y_{n+2}(1 + x^2) + n_{C_1}y_{n+1}(2x) + n_{C_2}y_n(2)] + 2\left(y_{n+1}x + n_{C_1}y_n 1\right) = 0$

$\Rightarrow y_{n+2}(1 + x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$

$\Rightarrow (1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0$ , $y_1(0) = 1$ and $y_2(0) = 0$

Also putting $x = 0$ in ④, we get

$y_{n+2}(0) = -n(n+1)y_n(0)$

Putting $n = 1,2,3$ ... in the above equation, we get

$$y_3(0) = -1(2)y_1(0)$$

$$= -2 \qquad \because y_1(0) = 1$$

$$y_4(0) = -2(3)y_2(0)$$

$$= 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = -3(4)y_3(0)$$

$$= -3(4)(-2) = 4!$$

$$y_6(0) = -4(5)y_4(0) = 0$$

$$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$$

$$\vdots$$

$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)!$ and $\quad y_{2n}(0) = 0$

**Example18** If $y = (sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

**Solution:** Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ......②

Squaring both the sides, we get

$$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$$

$$\Rightarrow (1-x^2){y_1}^2 = 4\,(y)^2$$

Differentiating the above equation w.r.t. $x$, we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$$

$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0$ ......③

Differentiating the above equation $n$ times w.r.t. $x$ using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n1) = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0\,, y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.\, 2^2.\, 4^2 \ldots\ldots\ldots.(n-2)^2, & if\ n\ is\ even \end{cases}$$

**Exercise 1 B**